Instructions

- You're welcome to work on these problems individually or in groups.
- If you have any questions, please don't hesitate to raise your hand and call me over! I'll be happy to help.

Algebraic Equations

Q1. *[Warm-up.]* Find all the solutions to the following equations.

(a) $x^2 - 9 = 0$.
(b) $x^2 - x - 2 = 0$.
(c) $x^2 + ax + a^2 = 0$, where a is a constant. (Express your solution in terms of a.)
(d) $x^3 - a^3 = 0$, where a is a constant. [**Hint:** What do you get if you multiply out $(x-a)(x^2+ax+a^2)$?]

Q2. *[A derivation of the quadratic formula.]* Consider the quadratic equation

$$ax^2 + bx + c = 0, \quad a \neq 0.$$

(a) Show that if we substitute $x = X + d$ into the equation then expand and simplify, we can turn it into

$$aX^2 + (2ad + b)X + (ad^2 + bd + c) = 0.$$

(b) By making an appropriate choice of d (which will depend on some of a, b, c), show that the equation in part (a) can be written in the form

$$aX^2 + p = 0.$$

Your p should be written in terms of a, b and c. [**Hint:** There is no X term in this equation. So try to get rid of the X term in (a)!]

(c) Solve your equation in part (b) for X, and therefore solve the original quadratic equation for x.

Q3. *[Vieta's relations.]*

(a) Let r_1 and r_2 be the roots of the quadratic equation $ax^2+bx+c = 0$. Prove the following expressions for the sum and product of the roots:

$$r_1 + r_2 = -\frac{b}{a} \quad \text{and} \quad r_1 r_2 = \frac{c}{a}.$$

[*Challenge:* Can you give two different proofs? Can you give more than two?]

(b) Vieta's relations tell you that you can "reconstruct" a quadratic equation if you know its roots. For example, can you find a quadratic equation whose roots are 1 and -3?

[**Hint:** To make your life easier, choose a quadratic equation where $a = 1$, i.e., an equation of the form $x^2 + bx + c = 0$.]

(c) Let $r_1,\ r_2$ and r_3 be the roots of the cubic equation $ax^3+bx^2+cx+d=0$. Discover and prove a relationship between the roots $r_1,\ r_2$ and r_3 and the coefficients $a,\ b,\ c$ and d of the cubic, similar to what we have in part (a). [**Hint:** Expand $a(x-r_1)(x-r_2)(x-r_3)$.]

Q4. *[Solving the depressed cubic.]* Consider the cubic equation

$$x^3 + px + q = 0. \qquad (*)$$

(In particular, there is no x^2 term: it's been "depressed"!)

(a) Let r be a root of $(*)$. For the moment, express r in the form $r = \alpha + \beta$, where α and β are unknown variables. Plug $x = r$ into $(*)$ then expand and simplify to show that

$$\alpha^3 + \beta^3 + (\alpha+\beta)(3\alpha\beta + p) + q = 0.$$

(b) Assume now that $\alpha\beta = -p/3$. Show that

$$\alpha^3\beta^3 = -\frac{p^3}{27} \quad \text{and} \quad \alpha^3 + \beta^3 = -q.$$

[**Hint:** For the second one, look at what you have in part (a).]

(c) Using part (b) and Vieta's relations from the previous problem, find a quadratic equation whose roots are α^3 and β^3. Make it so that the coefficients of your quadratic equation depend only on p and q.

(d) Solve your quadratic equation in part (c) to get expressions for α^3 and β^3.

(e) Use part (d) to find expressions for α and β, and hence for the root r. [*Note:* This gives you **Cardano's formula**.]

Q5. Consider the depressed cubic

$$x^3 + x - 2 = 0.$$

(a) Show that $x = 1$ is a root of the equation.

(b) Apply Cardano's formula from **Q4** to this equation. Do you get the root 1?
You might want to plug your formula into a calculator, such as Wolframalpha.
[*Note:* To enter square roots and cube roots into Wolframalpha, use `sqrt()` and `cbrt()`. So, for example, enter `sqrt(2) for` $\sqrt{2}$ `and cbrt(10) for` $\sqrt[3]{10}$.]

Q6. *[Solving the general cubic.]* Consider the cubic equation

$$ax^3 + bx^2 + cx + d = 0, \quad a \neq 0.$$

By dividing through by a, we can (and will) assume that $a = 1$, so our equation takes the simpler form

$$x^3 + bx^2 + cx + d = 0. \quad (**)$$

(a) Substitute $x = X + e$ into equation $(**)$ and simplify.

(b) Pick a choice of e that turns your equation in part (a) into a depressed cubic.

(c) Use Cardano's formula, from **Q4**, to solve $(**)$.

Note: The same kind of method can be applied to solve the quartic equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

but the algebra involved is fairly messy. You start by making a substitution to eliminate the x^3 term, creating a "depressed quartic." Then there is a very clever trick, discovered by L. Ferrari (one of Cardano's students), that allows us to complete the square in this "depressed quartic" and reduce our problem to a problem of solving a related cubic equation, which we know how to do!

The very strange thing is: these kinds of tricks fail when it comes to equations of degree 5 and higher, but it's not at all clear why...

What makes degree ≤ 4 special?!